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Refocusing of a qubit system coupled to an oscillator.

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Abstract. Refocusing, or dynamical decoupling, is a coherent control technique where the internal dynamics of a quantum system is effectively averaged out by an application of specially designed driving fields. The method has originated in nuclear magnetic resonance, but it was independently discovered in atomic physics as a "coherent destruction of tunneling". Present work deals with the analysis of the performance of "soft" refocusing pulses and pulse sequences in protecting the coherence of a qubit system coupled to a quantum oscillator.

Introduction and background. Quantum coherent control has found way into many applications, including nuclear magnetic resonance (NMR), quantum information processing (QIP), spintronics, atomic physics, etc. The simplest control technique is dynamical decoupling (DD), also known as refocusing, or coherent destruction of tunneling. The goal here is to preserve coherence by averaging out the unwanted couplings. This is achieved most readily by running precisely designed sequences of specially designed pulses [1].

In a closed system this can be analyzed in terms of the average Hamiltonian theory in the "rotating frame" defined by the controlling fields [2]. To leading order, the net evolution over the refocusing period is indeed described by the Hamiltonian of the system averaged over the controlled dynamics. For an open system, the dynamics associated with the bath degrees of freedom can be also averaged out, as long as they are sufficiently slow. This can be understood by noticing that the driven evolution with period $\tau = 2\pi/\Omega$ shifts some of the system's spectral weight by the Floquet harmonics, $\omega \to \omega + n\Omega$. With the average Hamiltonian for the closed system vanishing, the original spectral weight at $n = 0$ disappears altogether, and the direct transitions with the bath degrees of freedom are also suppressed as long as Ω exceeds the bath cut-off frequency, $\Omega \gtrsim \omega_c$ [3].

For a *closed* system, the refocusing can be made more accurate by designing higher-order sequences, where not only the average Hamiltonian ($k = 1$), but all the terms of order $k \le K$ in the Magnus (cumulant) expansion of the evolution operator over the period τ are suppressed. The corresponding simulation can be done efficiently by constructing time-dependent perturbation theory on small clusters [4]. The quantum kinetics of the corresponding *open* system with order-K DD, $K \le 2$, was analyzed by one of the authors using the non-Markovian master equation in the rotating frame defined by the refocusing fields [5]. This involved a resummation of the series for the Laplace-transformed resolvent of the master equation near each Floquet harmonic, with subsequent summation of all harmonics.

The results of Ref. [5] can be summarized as follows. With $K \ge 1$ refocusing, there are no direct transitions, which allows an additional expansion in powers of the small adiabaticity parameter, ω_c/Ω. In this situation the decoherence is dominated by reactive processes (dephasing, or phase diffusion). With $K = 1$, the bath correlators are modulated at frequency Ω. This reduces the effective bath correlation time, and the phase diffusion rate is suppressed by a factor $\propto \omega_c/\Omega$. With $K = 2$ refocusing, all 2nd-order terms involving instantaneous correlators of the bath coupling are cancelled. Generically, this leads to a suppression of the dephasing rate by an additional factor $\propto (\omega_c/\Omega)^2$, while in some cases (including single-qubit refocusing) all terms of the expansion in powers of the small adiabaticity parameter (ω_c/Ω) disappear. This causes an *exponential* suppression of the dephasing rate, so that an excellent refocusing accuracy can be achieved with relatively slow refocusing, $\Omega \gtrsim \omega_c$.

The conclusions in Ref. [5] were based on the analysis of the oscillator bath with a featureless spectral function, with the cut-off frequency ω_c serving as the only scale describing the bath correlations. They do not apply in the presence of sharp spectral features which appear if the controlled system is coupled to a local high-Q oscillator. On the other hand, the situation where the controlled system is coupled to a local oscillator mode is quite common. This situation is realized in atomic physics, where the oscillator in question is the cavity mode, while the continuous-wave (CW) excitation is used to suppress the coupling. In several quantum computer designs, nearly-linear oscillator modes are inherently present (e.g., mutual displacement in ion traps, or QCs based on electrons on helium). Finally, there are suggestions to include local high-Q oscillators in the QC designs to serve as "quantum memory" or "quantum information bath" [6].

In this work we consider refocusing of a qubit system where the spectral function of the oscillator bath has a sharp resonance. More specifically, we include the resonant mode in the system Hamiltonian, and consider the quantum kinetics of the resulting system in the presence of a featureless low-frequency oscillator bath driven by the refocusing pulses applied to the qubits only. Such a system can be analyzed with the help of the general results [5], as long as one is able to construct a $K = 1$ or $K = 2$ refocusing sequence to decouple the oscillator and other degrees of freedom. To this end, and having in mind sequences of soft pulses, we consider the analytical structure of the evolution operator for a closed system of arbitrary complexity, where one of

the qubits is driven by a single one-dimensional π-pulse. An analysis of any refocusing sequence is then reduced to computing an ordered product of evolution operators for individual pulses. We illustrate the technique by analyzing the controlled dynamics of a qubit coupled to an oscillator. One of the constructed sequences provide order $K = 2$ qubit refocusing for any form of qubit–oscillator coupling, and was also shown to provide an excellent decoupling in the presence of a thermal bath.

Single π-pulse. Consider a qubit with generic couplings,

$$H_S = \sigma_x A_x + \sigma_y A_y + \sigma_z A_z + A_0, \tag{1}$$

where σ_μ are the qubit Pauli matrices and A_ν, $\nu = 0, x, y, z$ are the operators describing the degrees of freedom of the rest of the system which commute with σ_μ, $[\sigma_\mu, A_\nu] = 0$. The qubit evolution is driven by a one-dimensional pulse,

$$H_C = \tfrac{1}{2}\sigma_x V_x(t), \quad 0 < t < \tau_p, \tag{2}$$

where the field $V_x(t)$ defines the pulse shape. The evolution due to the pulse is dominant; the unitary evolution operator to zeroth order in H_S is simply

$$U_0(t) = e^{-i\sigma_x \phi(t)/2}, \quad \phi(t) \equiv \int_0^t dt'\, V_x(t'). \tag{3}$$

When acting on the spin operators, this is just a rotation, e.g., $U_0(t)\sigma_y U_0^\dagger(t) = \sigma_y \cos\phi(t) + \sigma_z \sin\phi(t)$. Suppose $V_x(t)$ be symmetric, $V_x(\tau_p - t) = V_x(t)$, π-pulse, $\phi(\tau_p) = \pi$, and it additionally satisfies the first-order self-refocusing condition $s \equiv \langle \sin\phi(t)\rangle_p = 0$, where $\langle f(t)\rangle_p$ denotes the time-average over pulse duration. Then, the evolution operator $X \equiv U_0(\tau_p)$ expanded to second order in $\tau_p H_S$ reads

$$\begin{aligned} X^{(2)} &= -i\sigma_x - \tau_p(A_x + \sigma_x A_0) + \frac{i}{2}\tau_p^2\{A_0, A_x\} \\ &+ \frac{i}{2}\tau_p^2 \sigma_x (A_0^2 + A_x^2) + \tau_p^2 \alpha\big(A_y^2 + A_z^2 + i\sigma_x[A_y, A_z]\big) \\ &+ \tau_p^2 \zeta\big([A_0, \sigma_y A_z - \sigma_z A_y] + i\{A_x, \sigma_y A_y + \sigma_z A_z\}\big). \end{aligned} \tag{4}$$

Here $\alpha \equiv \langle \theta(t-t')\sin[\phi(t)-\phi(t')]\rangle_p$, $\zeta \equiv \langle \theta(t-t')\cos\phi(t')\rangle_p$ parametrize the evolution properties of the pulse at second order. The values of the parameters computed for some pulse shapes are listed in Tab. 1.

Common pulse sequences. Transforming Eq. (4) appropriately, we can now easily compute the result of application of any pulse sequence. In particular, the π-pulse $\overline{X}$ applied along the $-x$ direction can be obtained from $(-X)$ with a substitution $\alpha \to -\alpha$. As a result, e.g., the expansion of the evolution opeator for the one-dimensional sequence $\overline{X}X$ can be written as

$$\overline{X}X = 1 - 2i\tau_p(A_0 + \sigma_x A_x) - 2\tau_p^2(A_0 + \sigma_x A_x)^2 + \mathcal{O}(\tau_p^3),$$

or it can be re-exponentiated as evolution with the effective Hamiltonian

$$H_{\overline{X}X} = A_0 + \sigma_x A_x + \mathcal{O}(\tau_p^2). \tag{5}$$

pulse	$s \equiv \langle \sin\phi(t)\rangle_p$	$\alpha/2$	ζ
$\pi\delta(t-\tau_p/2)$	0	0	0.25
G_{001}	0.0148978	0.00735798	0.249979
G_{010}	0.148979	0.0653938	0.247905
S_1 [4]	0	0.0332661	0.238227
S_2 [4]	0	0.0250328	0.241377
Q_1 [4]	0	0	0.239889
Q_2 [4]	0	0	0.242205

Table 1. Parameters of several common pulse shapes. The first line represents the "hard" δ-function pulse, G_{001} denotes the Gaussian pulse with the width $0.01\tau_p$, while S_n and Q_n denote the 1st and 2nd-order self-refocusing pulses from Ref. [4].

The corresponding calculation with the usual finite-width pulses (e.g., Gaussian) where $s \neq 0$ produces a correction to the effective Hamiltonian already in the leading order,

$$\delta H_{\overline{X}X} = s(\sigma_z A_y - \sigma_y A_z) + \mathcal{O}(s^2\tau_p).$$

This can be corrected by constructing a longer sequence, e.g., $\overline{X}XX\overline{X}$. Returning to pulses with $s = 0$, we list the expansions computed for several two-dimensional sequences:

$$\begin{aligned} H_{X\overline{Y}XY} &= A_0 + \tau_p\Big(\frac{i\alpha}{2}[A_z, A_y] - \frac{i}{2}[A_0, \sigma_x A_x - \sigma_y A_y] \\ &\quad - \frac{\alpha}{2}\sigma_y(A_x^2 + A_y^2) + \frac{1+4\zeta}{4}\sigma_z\{A_x, A_y\}\Big) + \mathcal{O}(\tau_p^2), \qquad (6) \\ H_{YX\overline{Y}XX\overline{Y}XY} &= A_0 - \frac{\alpha\tau_p}{2}\Big(\sigma_y(A_x^2 + A_z^2) + i[A_y, A_z]\Big), \qquad (7) \\ H_{\overline{Y}\overline{X}Y\overline{X}X\overline{Y}XY} &= A_0 + \mathcal{O}(\tau_p^2). \qquad (8) \end{aligned}$$

Atom in a cavity example. Consider an atom placed in a lossless cavity with a single resonant mode. The resonance part of the Hamiltonian can be written in the form (1), where $A_x = g(b + b^\dagger)$, $A_y = ig(b - b^\dagger)$, $A_z = 0$, $A_0 = \Delta\, b^\dagger b$, g is a coupling constant, and Δ is the cavity frequency bias. Contrary to the conclusions of Ref. [7], such a coupling cannot be suppressed with any one-dimensional pulse sequence.

On the other hand, the two-dimensional four-pulse sequence (6) provides a leading-order refocusing of the coupling. The subleading-order correction is present (the order of the sequence is $K = 1$), and it is not particularly small even for 2nd-order self-refocusing pulses with $s = \alpha = 0$. The eight-pulse sequences (7) and (8) have equal accuracy with 2nd-order pulses but the 2nd-order accuracy of the latter sequence is also retained with 1st-order pulses.

After tracing out the oscillator degrees of freedom, we can apply the results of Ref. [5] and expect the two 2nd-order sequences to provide an excellent refocusing accuracy even in an open system, as long as the refocusing rate is sufficiently high. We confirmed this expectation by a numerical simulation, where the bath was modeled as a classical correlated random field.

Conclusions. The main result of this work is the expansion (4) and the classification of the corresponding parameters in Tab. 1. This allows an explicit computation of the

error operators associated with refocusing in systems of arbitrary complexity. We illustrated the approach for several sequences applied to a qubit coupled to an oscillator. The 8-pulse sequence (8) provides 2nd-order refocusing for any form of the coupling between the qubit and the oscillator.

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References

[1] C. P. Slichter, Principles of Magnetic Resonance (Springer-Verlag, Berlin, 1992), 3rd ed.; L. M. K. Vandersypen and I. L. Chuang, Rev. Mod. Phys. **76**, 1037 (2005).

[2] J. S. Waugh, L. M. Huber, and U. Haeberlen, Phys. Rev. Lett. **20**, 180 (1968); J. S. Waugh, C. H. Wang, L. M. Huber, and R. L. Vold, J. Chem. Phys. **48**, 662 (1968).

[3] A. G. Kofman and G. Kurizki, Phys. Rev. Lett. **87**, 270405 (2001); **93**, 130406 (2004).

[4] P. Sengupta and L. P. Pryadko, Phys. Rev. Lett. **95**, 037202 (2005).

[5] L. P. Pryadko and P. Sengupta, Phys. Rev. B **73**, 085321 (2006).

[6] See, e.g., A. T. Sornborger, A. N. Cleland, and M. R. Geller, Phys. Rev. A **70**, 052315 (2004) and references therein.

[7] M. Thorwart, L. Hartmann, I. Goychuk, P. Hänggi, J. Mod. Opt. **47**, 2905 (2000).